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**LETTER TO THE EDITOR**

# The Ising model in a random field; supersymmetric surface fluctuations and their implications in three dimensions

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**Abstract.** The effective energy for interfaces between the ferromagnetic phases of the Ising model in a random external field is obtained by exploiting the supersymmetry of the model. This effective energy in three dimensions has the same renormalisation properties as the one-dimensional pure Ising model, to all orders in Feynman graph perturbation theory. On the basis of this analogy we present predictions for the correlation functions of the random field model in three dimensions.

The aim of this paper is to indicate the role of supersymmetry in controlling interface fluctuations in the Ising model in a random external field. Our main interest is to study how the existence of a phase transition depends on the spatial dimension of the system. The basic approach we adopt is to study whether for an arbitrarily weak random external field the model can sustain a local interface separating the two distinct ferromagnetic low-temperature phases. In this approach it is crucial to incorporate properly the statistical mechanics of the gapless capillary waves, whose origin can be traced to the breaking of rotation and translation symmetry by e.g. a planar interface (Wallace and Zia 1979, Lowe and Wallace 1980). For the Ising model in a random field we shall see that a ferromagnetic interface breaks not only these symmetries of a continuum model but also the supersymmetry of the model exposed by Parisi and Sourlas (1979). Consequently there are important gapless fluctuations in the random field problem which are in addition to the standard capillary waves of the interface in a pure system. We shall show how these effects combine to raise the lower critical dimension of the random model to 3.

This problem has been studied in a recent Letter by Pytte *et al* (1981), hereafter referred to as I. Our picture for the phase diagram agrees qualitatively with that obtained in I by means of the replica method (Edwards and Anderson 1975). However, we are unable to understand the model for surface fluctuations proposed in I and in particular, calculations on the two, four and six point correlation functions show that the model used in I for calculation is not consistently renormalisable in a sense we shall expand upon below. It seems likely to us that terms omitted in calculations in I are relevant, despite contrary claims by the authors.

Our starting point is the conventional Landau–Ginzburg–Wilson model for this problem—a one-component field $\phi(x)$ in a random external field $h(x)$:

$$\mathscr{H} = \int d^d x \, [\tfrac{1}{2}(\nabla\phi(x))^2 + V[\phi(x)] + h(x)\phi(x)]. \tag{1}$$

For a *quenched* random field $h$, the prescription for calculating e.g. the free energy involves averaging the free energy for a given $h(x)$ over all functions $\{h\}$ with a probability distribution whose simplest form is $P\{h\} \propto \exp[-\frac{1}{2}\int d^dx\, h^2(x)/\Delta]$. Clearly the parameter $\Delta$, which will play an important role subsequently, measures the strength of the randomness by controlling the width of the distribution of $h$.

The study of the model (1) by renormalisation group methods shows that non-trivial critical behaviour can be obtained as a power series in $\varepsilon$ in $6$-$\varepsilon$ dimensions (Imry and Ma 1975, Lacour-Gayet and Toulouse 1974, Grinstein 1976, Aharony *et al* 1976) and that the perturbation expansions for exponents are the same to all orders in $\varepsilon$ as the expansions for the pure Ising-type model in $4$-$\varepsilon$ dimensions (Young 1977). An extremely elegant way to establish this perturbative equivalence was pointed out by Parisi and Sourlas (1979). They showed that the diagrams contributing to the calculated critical behaviour could be expressed in terms of conventional scalar fields $\phi(x)$, $\omega(x)$ and anticommuting scalar fields $\psi(x)$, $\bar{\psi}(x)$. The interactions amongst these fields involve only the potential $V(\phi)$ of (1) and can be written in a simple form if we collect the fields together into a superfield (Salam and Strathdee 1974, 1975, Delbourgo 1975, Fayet and Ferrara 1977) by using anticommuting variables $\theta$ and $\bar{\theta}$, defining

$$\Phi(x, \theta, \bar{\theta}) = \Delta^{-1/2}\phi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \theta\bar{\theta}\Delta^{1/2}\omega(x). \tag{2}$$

The Hamiltonian replacing (1) then takes the form

$$\mathscr{H}_{\mathrm{R}} = \int d^dx\, d\bar{\theta}\, d\theta[\tfrac{1}{2}\Phi(-\nabla_s^2\Phi) + \Delta^{-1}V(\Delta^{1/2}\Phi)] \tag{3}$$

where

$$\nabla_s^2 = \nabla^2 + \partial/\partial\bar{\theta}\,\partial/\partial\theta. \tag{4}$$

This looks superficially like the pure model analogous to (1) except that (a) the coefficient $g$ of the quartic coupling for example is replaced by $g\Delta$ and (b) the field now exists in a higher-dimensional space $(x, \theta, \bar{\theta})$. However, Parisi and Sourlas show that the anticommuting variables effectively *reduce* the spatial dimension $d$ by 2, i.e. this most singular part of the random system in $d$ dimensions is equivalent to the pure system in $d$-2 dimensions, at least to all orders in perturbation theory.

We now turn to the problem of describing an interface between two low-temperature ferromagnetic phases of (3). It is helpful to recall the arguments for the original pure model (Wallace 1980, Jasnow and Rudnick 1978, Ohta and Kawasaki 1977),

$$\mathscr{H} = \int d^dx\, [\tfrac{1}{2}\phi(-\nabla^2\phi) + V(\phi)]. \tag{5}$$

We assume that we are below a critical temperature on the coexistence curve, so that $V(\phi)$ has two minima $\phi_\pm$ and $V(\phi_+) = V(\phi_-)$, at the classical level; it is straightforward to generalise this condition when fluctuation effects are incorporated. The classical description of the interface then involves the solution $\phi_c(x)$ of the field equation extremising $\mathscr{H}$,

$$\nabla^2\phi = \partial V/\partial\phi, \tag{6}$$

obeying the boundary conditions $\phi_c \to \phi_\pm$ as $z \to \pm\infty$ say. A solution $\phi_c(z)$ depending only on the single coordinate $z$ can be found; it gives the classical density profile for a flat interface whose normal is along the $z$ axis. However, because the interface can be translated or rotated in this model at no energy cost, we know that in the fluctuations there will be modes which are gapless in the long-wavelength limit—the capillary waves. For the pure system we know that surface tension controls the effective energy of such fluctuations. We can derive this result by noting that an accurate enough description of an interface displaced by a perpendicular distance $f(y)$ from the surface $z=0$ is given by

$$\phi_f(z, y) = \phi_c\left(\frac{z-f(y)}{[1+(\nabla f)^2]^{1/2}}\right). \tag{7}$$

The numerator in the argument translates the interface locally by $f(y)$ depending on the $(d-1)$ component position $y$ and the denominator describes the apparent local dilation of the profile of an interface making a direction cosine $[1+(\nabla f)^2]^{-1/2}$ with the $z$ axis. An elementary change of variables then yields an effective energy for such a fluctuation

$$\mathscr{H}_{\text{eff}}(f) = \mathscr{H}(\phi_f) = \int \mathrm{d}z\ (\phi_c'(z))^2 \int \mathrm{d}^{d-1}y\ [1+(\nabla f)^2]^{1/2} \tag{8}$$

when we neglect higher derivatives of $f$ to obtain the long-wavelength limit. Here the coefficient of the area term is the mean-field approximation for the surface tension.

By applying standard renormalisation group methods to the model (8), there emerges (Wallace and Zia 1979) a systematic description of critical behaviour in Ising-type models near the lower critical dimension, where the capillary waves destroy even the existence of a local interface (i.e. in 1 and $1+\varepsilon$ dimensions). A crucial feature of the consistency of these calculations is that renormalisation of all correlation functions of $f$ is achieved by a renormalisation of the one available quantity, the prefactor in (8).

Stimulated by the paper of Pytte *et al* (I), we wish to consider the analogous problem for the random model (1). On the basis of previous experience with classical solutions in $n \to 0$ models (Houghton *et al* 1978), we are wary about the reliability of the replica method for such a problem; indeed with the model for surface fluctuations stated in I we obtain inconsistent results when we attempt to renormalise the $2n$ point functions by renormalisation of the available couplings. We prefer therefore to start from the model (3) exhibiting explicitly the supersymmetry. Inspection of the field equations reveals that $\Phi(x, \theta, \bar{\theta}) = \Delta^{-1/2}\phi_c(z)$ is an extremum of (3), where $\phi_c(z)$ is the standard interface solution of (6). In the supersymmetric model (3) this configuration breaks not only some of the rotations and translations but also the supersymmetry. A little reflection should convince the reader that a sensible generalisation of (7) for the superfield configuration corresponding to a displaced interface is

$$\Phi(x, \theta, \bar{\theta}) = \Delta^{-1/2}\phi_c\left(\frac{z-F(y, \theta, \bar{\theta})}{[1+(\nabla F)^2+(\partial F/\partial\bar{\theta})\,\partial F/\partial\theta]^{1/2}}\right). \tag{9}$$

This guess is sufficient to derive in a controlled fashion (Wallace 1980) the effective Hamiltonian† for the soft modes at fixed low temperature:

$$\mathscr{H}_{\text{eff}}(F) = \tilde{\Delta}^{-1} \int \mathrm{d}^{d-1}y\ \mathrm{d}\bar{\theta}\ \mathrm{d}\theta\ [1+(\nabla F)^2+(\partial F/\partial\bar{\theta})\,\partial F/\partial\theta]^{1/2} \tag{10}$$

† Technically, the symmetry group of the superspace model (3) is a contraction of $\mathrm{OSp}(d+1, 2)$ with the $\mathrm{O}(d+1)$ subgroup contracted to the Euclidean group of rotations and translations in $d$ dimensions. The coordinates $(x, \theta, \bar{\theta})$ in (3) transform as the coset space of this group factored by $\mathrm{OSp}(d, 2)$. The nonlinear transformation of the field $F(y, \theta, \bar{\theta})$ in the interface model (10) is determined by the transformation of the surface $z = F(y, \theta, \bar{\theta})$ in the superspace $(x, \theta, \bar{\theta})$.

where (cf (8)) $\tilde{\Delta}^{-1} = \Delta^{-1} \int dz\, (\phi_c'(z))^2$. A scaling ansatz for this coefficient in three dimensions would suggest

$$\tilde{\Delta} \propto \Delta (T_c - T)^{-2\nu}, \qquad \nu \cong 0.63, \tag{11}$$

where $T_c$ is the transition temperature in the absence of the external field.

An attractive feature of this formalism is that the Parisi and Sourlas argument now implies that any results from (8) near one dimension may be carried over to results for (10) near three dimensions. In particular, the Hamiltonian (10) is renormalisable in three dimensions by a renormalisation of $\tilde{\Delta}$ alone, to all orders in perturbation theory. Corresponding to the $T = 0$ critical point of the 1-$d$ Ising model with a correlation length $\xi \propto T^{1/2} \exp(\text{constant}/T)$, the model (10) has no ferromagnetic phase transitions for any finite $\tilde{\Delta}$, the correlation length diverging in three dimensions as

$$\xi \propto \tilde{\Delta}^{1/2} \exp(1/\tilde{\Delta})[1 + O(\tilde{\Delta})]. \tag{12}$$

We have verified these results also by explicit calculation to two loops. In three dimensions our picture then is a phase diagram in the $\Delta T$ plane consisting of a line of critical points for $\Delta = 0$, $T < T_c$.

Of course the limitations of these remarks should be apparent. We have calculated only the effect of the most singular diagrams, on the presumption that the others will remain corrections to scaling in $d = 3$. This study of the stability of a ferromagnetic interface says nothing about the possibility of ordering into e.g. a spin-glass phase (Morgenstern *et al* 1981, De'Bell 1981) (although we believe that a model analogous to (10) for almost spherical droplets will yield a good description of a 'domain structure' without a phase transition (Binder *et al* 1981) for small $\Delta$). In the absence of a theorem that all planar interface configurations for (10) are equivalent to $\phi_c(z)$, we cannot even rule out with any rigour the stability of a totally different ferromagnetic interface configuration. We note that the results (10)–(12) are not in general linked with the Griffiths singularity (Griffiths 1969, Imry 1977) because of the totally different dependence on dimensionality.

These qualifications notwithstanding, let us now make the bold extrapolation of lifting into three dimensions the scaling form of the correlation function of the one-dimensional Ising model: $\langle \phi(x)\phi(0)\rangle = \exp[-(|x|/\xi)]$. This suggests the correlation function $\langle \Phi(x, \theta, \bar{\theta})\Phi(0)\rangle = \exp[-(x^2 + 4\theta\bar{\theta})^{1/2}/\xi]$ for the random field model (3) in three dimensions. Now, neutron scattering experiments (Cowley *et al* 1981) on a dilute Ising antiferromagnet in a uniform external field $H$ measure the quenched two-point correlation function (Fishman and Aharony 1979) of the model (1), with the identification $\Delta \propto H^2$. Extending the argument of Parisi and Sourlas (1979), the most singular part of this measured correlation function is obtained directly from the superfield correlation function and, after Fourier transform, takes the form

$$G(q) = 8\pi\xi^{-1}[\Delta(\xi^{-2} + q^2)^{-2} + (\xi^{-2} + q^2)^{-1}]. \tag{13}$$

Several features of this expression are apparent in the neutron scattering data (Cowley *et al* 1981) on $Co_xZn_{1-x}F_2$. We note particularly that as the applied field $H \to 0$, $\Delta \propto H^2 \to 0$ also. The range in $q$ values where the Ornstein–Zernike term $(\xi^{-2} + q^2)^{-1}$ in (13) gives a significant contribution is $q \leqslant \xi^{-1/2}$, which according to (11) and (12) shrinks rapidly as $H \to 0$. Moreover the first term in (13) then becomes more important, by a factor $\sim \Delta\xi \to \infty$. Thus as $H$ decreases it is possible to obtain crossover from an Ornstein–Zernike form to a more singular one, as required by the data. Further, the form (12) implies that $\xi^{-1}$ is vanishingly small over a range of small $H$;

indeed, if we take the form of $\xi$ from the one-dimensional model it also has for intermediate $H$ a quasi-linear dependence on $H$, in quantitative agreement with the data. With this measure of agreement, it seems that the clean description of the interface above is likely to provide a useful framework within which to understand the experimental results.

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